MATHEMATICS

ADDENDUM TO "THE ABSTRACT RIEMANN INTEGRAL AND A THEOREM OF G. FICHTENHOLZ ON EQUALITY OF REPEATED RIEMANN INTEGRALS IA AND IB"

BY

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Professor Ridder brought to my attention that there is an error in my paper "The abstract Riemann integral and a theorem of G. Fichtenholz on equality of repeated Riemann integrals IA and IB" (Nederl. Akad. Wetensch. Proc. ser. A64 = Indag. Math. 23, 516–545 (1961)). The error occurs in section 2 "Counterexamples", where, in the paragraph starting from line 10 from the top of page 518, proposition C_{49} of Sierpinski's "Hypothèse du Continu" is misquoted. In fact, Professor Ridder pointed out to me that it is possible to replace Riemann integrability by Lebesgue integrability in either condition (ii) or (iii) but not in both. In this form, the theorem is due to L. LICHTENSTEIN (see [2]). It implies immediately the theorem of Fichtenholz. From the date of the paper of Lichtenstein it seems to be more appropriate to speak about the theorem of Fichtenholz–Lichtenstein in the case one restricts integration to the domain of the Riemann integral.

It may be of interest to the reader to point out that the theorem of Lichtenstein can be generalized to the abstract case as well. Using the notation and terminology of my paper, in particular of sections 8 and 9, quoted in the first paragraph, we shall prove the following generalization of the theorem of Lichtenstein.

Let X and Y be two non-empty point sets, and let Γ_x and Γ_y be semirings of subsets of X and Y respectively. Furthermore, we assume that ν_x and ν_y are signed measures defined on Γ_x and Γ_y respectively and that $X \in (\Gamma_x)_s$ and $Y \in (\Gamma_y)_s$, $|\nu_x|(X) \neq 0$ and $|\nu_y|(Y) \neq 0$. We shall denote the Riemann integral with respect to ν_x by J_x and the Lebesgue integral with respect to ν_y by L_y respectively.

Theorem (L. Lichtenstein). *Let Ω_x be a collection of increasing sequences of partitions of X. If $f = f(x, y)$, $(x, y) \in X \times Y$, is a real function satisfying the following conditions:*

(i) *for all $(x, y) \in X \times Y$, we have $|f(x, y)| \leqslant M$,*

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(ii) *for all* $y \in Y$, $f(x, y) = f_y(x)$ $(x \in X)$ *is* Ω_{x,ν_x}*-integrable over* X,

(iii) *for all* $x \in X$, $f(x, y) = f_x(y)$ $(y \in Y)$ *is* ν_y*-Lebesgue integrable over* Y,

then $\phi(x) = L_y(f_x)$ $(x \in X)$ *is* Ω_{x,ν_x}*-integrable over* X *and* $\psi(y) = J_x(f_y)$ $(y \in Y)$ *is* ν_y*-Lebesgue integrable over* Y *respectively, and* $J_x(\phi) = L_y(\psi)$, *i.e., the repeated integrals* $J_x(L_y(f))$ *and* $L_y(J_x(f))$ *exist and are equal.*

Proof. We shall prove that ϕ is Ω_{x,ν_x}-integrable over X. For this purpose, let $v(\phi)$, $v'(\phi) \in VS(\phi)$ and let $A \in \Gamma_x$. For every partition π of X we have $J_x(v(\phi, \pi)) = J_x(v(L_y(f_x), \pi)) = J_x(L_y(v(f_x, \pi))) = L_y(J_x(v(f_x, \pi)))$. Hence, if $\omega_x \in \Omega_x$, then for all $n \in N$, we have

$J_x\{(v(\phi, \pi_{\omega_x, n}) - v'(\phi, \pi_{\omega_x, n}))\chi_A\} = L_y(J_x[\{v(f_y, \pi_{\omega_x, n}) - v'(f_y, \pi_{\omega_x, n})\}\chi_A]) = L_y(g_n)$. From (ii) it follows that $\lim_{n\to\infty} g_n(y) = 0$ for all $y \in Y$. Furthermore, $|g_n| \leqslant 2M|\nu_y|(Y)$ $(n \in N)$; and hence, by Lebesgue's dominated convergence theorem, $\lim_{n\to\infty} L_y(g_n) = 0$, i.e., ϕ is Ω_{x,ν_x}-integrable over X.

In order to prove that ψ is ν_y-Lebesgue integrable over Y, we observe that $\psi(y) = \lim_{u\to\infty} h_n(y)$ $(y \in Y)$, where $h_n(y) = J_x\{v(f_y, \pi_{\omega_x, n})\}$, $n \in N$ and $y \in Y$, $v \in VS$. Since $|h_n| \leqslant M|\nu_y|(Y)$ it follows that ψ is the limit of a bounded sequence of step functions. Hence, ψ is ν_y-Lebesgue integrable over Y and Lebesgue's dominated convergence theorem implies that $L_y(\psi) = \lim_{n\to\infty} L_y(h_n)$. From $L_y(h_n) = J_x(v(\phi, \pi_{\omega_x, n}))$ and ϕ is Ω_{x,ν_x}-integrable it follows finally that $J_x(\phi) = \lim_{n\to\infty} J_x(v(\phi, \pi_{\omega_x, n})) = \lim_{n\to\infty} L_y(h_n) = L_y(\psi)$. This completes the proof of the theorem.

Remarks. 1. The theorem is symmetric in x and y.

2. The abstract Riemann-integral version of the theorem of Fichtenholz-Lichtenstein follows immediately from the above theorem. Indeed, Ω-integrability implies Lebesgue integrability.

Professor Kemperman drew my attention to Lemma 2.3 of [1], which is a special case of the abstract Lichtenstein theorem.

We have given below a more complete list of references, which we owe to Professor Ridder, concerning the theorems of Fichtenholz and Lichtenstein. In [9], the reader will find a complete discussions of possible counter-examples. In fact, it makes section 2 of my original paper superfluous.

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